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DECOUPLING AND ORDER REDUCTION FOR LINEAR TIME-VARYING TWO-TIME--ETC(U)

JUL 80 R E O'MALLEY, L R ANDERSON N00014-76-C-0326

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Decoupling and Order Reduction for Linear Time-Varying

Two-Time-Scale Systems

Robert E. O'Malley, Jr.
Department of Mathematics and Program in Applied Mathematics
University of Arizona
Tucson, AZ 85721

and

Leonard R. Anderson
Aerospace and Ocean Engineering Department
Virginia Polytechnic Institute and State University
Blacksburg, VA 24061

Abstract

The class of time-varying linear systems which are two-time-scale on an interval may be decoupled by a time-varying transformation of variables into separate subsystems containing the slow and fast dynamic parts. The transformation is obtained by solving a nonsymmetric Riccati differential equation forward in time and a linear matrix differential equation backward in time. Small parameters are identified which measure the strength of the time scale separation and the stability of the fast subsystem. As these parameters go to zero, the order of the system is reduced and a useful approximate solution to the original system is obtained. The transformation is illustrated for examples with strong and weak fast subsystem stability.

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1. Motivating examples of time-invariant problems.

The longitudinal dynamics of an F-8 aircraft (cf. Etkin [12] and Teneketzis and Sandell [35]) can be modeled by an initial value problem for a fourth order linear system of the form $\dot{x} = Ax + Bu$ with the physical variables being the "primarily slow" velocity variation and flight path angle and the "primarily fast" angle of attack and pitch rate. The single control is the elevator deflection. The exact solution for the free response of the components of $x(t)$ is plotted in Figures 1 and 2. Our objective is to determine a solution $\hat{x}(t)$ of a reduced second order model which will approximate the dynamics of the given fourth order model away from the initial time $t = 0$. We wish, in particular, to avoid integration of the full order system or a complete eigenanalysis of A which would provide the exact solution $x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}B(s)u(s)ds$. We note that approximations to the matrix exponential e^{At} are, indeed, still the subject of current research (cf. Moler and Van Loan [23]) and that they are not simple to compute.

Our criteria for such approximations will naturally involve the eigenvalues of A. For this problem, we have the "slow" eigenvalues $s_{1,2} = -0.0075 \pm i0.076$ and the "fast" eigenvalues $f_{1,2} = -0.94 \pm i3.0$. Our method will rely upon the time-scale separation, measured by the smallness of the parameter $\mu = |s_2/f_1| = 0.024$, and the relative stability of the fast eigenvalues, measured by the parameter $\sigma = -|\text{Re } s_2|/\text{Re } f_1 = 0.0081$. Most important, however, is that the ratio of the fast-mode decay constant $(-1/\text{Re}(f_1))$ to the length T of the time interval of interest satisfies

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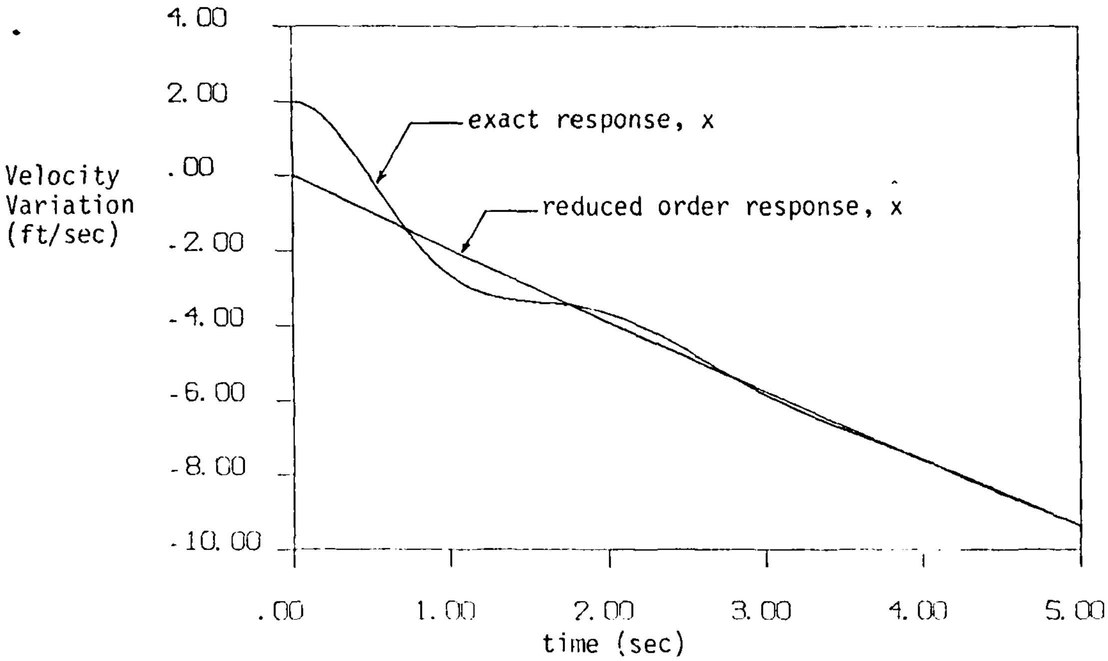

Figure 1:

F-8 aircraft mod
Velocity variati
(a slow variable
vs. time

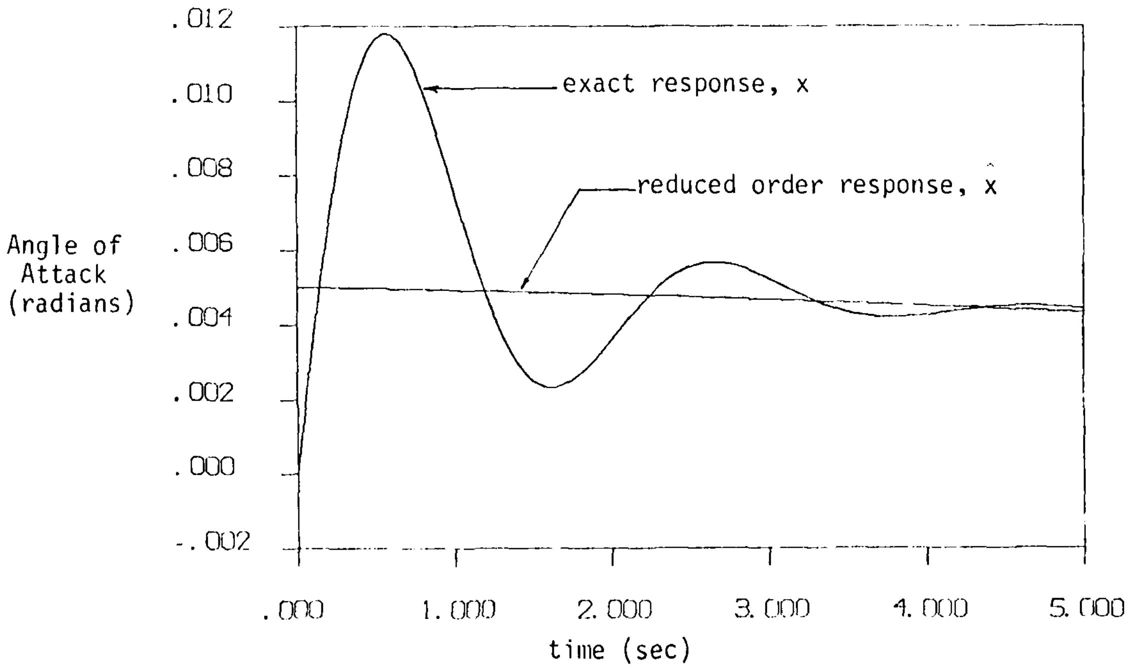

Figure 2:

F-8 aircraft mod
Angle of attack
(fast variable)
vs. time.

$$0 < \varepsilon = \frac{-1}{\mathrm{Re}(f_1)T} = \frac{1.06}{T} << 1 .$$

For μ and ε small we expect our reduced order model to be a good approximation to the solution on an interval $O(\varepsilon T) < t < T$, while on $0 < t < O(\varepsilon T)$ any fast mode components excited by the initial conditions may be significant and the approximation $\hat{x}(t)$ which ignores them would be inappropriate. As the figures suggest, T must be quite large in order for the initial layer to be relatively narrow.

For large dimensional linear problems, one cannot readily compute exact solutions to compare approximate solutions against. In power system models, systems involving several hundred variables are common. They are often approximated by reduced order models involving both differential and nonlinear algebraic equations which neglect fast initial transients (cf. Van Ness [37]). An algebraic system $g(x,z,t) = 0$ could correspond to a steady-state for the differential system $\varepsilon\dot{z} = g(x,z,t)$ with the small parameter ε representing "parasitics". The practical importance of obtaining reduced order models follows largely because the computational effort involved in numerically integrating systems of differential equations increases at least as the square of the order.

A second example of a two-time-scale problem is the sixteenth order model of a turbofan engine which was the theme problem for a recent conference on control of linear multi-variable systems (cf. Sain [31], Skira and DeHoff [33], and DeHoff and Hall [10]). The linear model is of the form $\dot{x} = Ax + Bu$, $y = Cx + Du$, with the state variables being fan speeds, pressures and temperatures. The five control variables u are fuel flow, exit nozzle area, two vane angles, and compressor bleed; and the outputs y are thrust, total airflow, a temperature, and two stall margins. The objective in the controller design is to achieve rapid thrust response without violating several operating constraints. The model is one of thirty-six different linear models obtained from a non-linear simulation of the engine. It represents the turbofan operating at sea level with near maximum non-afterburner power. This is an operating point which every engine must pass through at takeoff. Based on the eigenvalues of this model and $T = 2$ the time-scale separation and fast mode stability parameters are $(\mu,\varepsilon) = (0.304, 0.000867)$, $(0.371, 0.0285)$, $(0.383, 0.0744)$, respectively, for the number n_1 of "slow" modes chosen as 15, 5 or 3. Since an order reduction from 16 to 15 isn't substantial, we shall use $n_1 = 5$. In all cases, the time scale separation and relative stability parameters μ and σ are only marginally small, while the fast-mode stability parameter ε is quite small. We nonetheless obtain good approximate solutions by solving a reduced (fifth) order system instead of the original sixteenth order problem. In Figures 3 and 4, the exact solution of the sixteenth order problem and the solution of our fifth order model of the slow dynamics are plotted for the thrust and fan speed in response to changes in controls. The control inputs are step changes in fuel flow rates and inlet guide vane position. The second case, cf. Figure 4, provides a severe test to the reduced order model since the inlet guide vane is located at the front end of the engine and there is some delay before its effect is propagated to the net thrust. We note that the approximations are not good for $t < 0.28 \approx 10\varepsilon$ and that this initial layer will become narrower relative to T as ε tends toward zero.

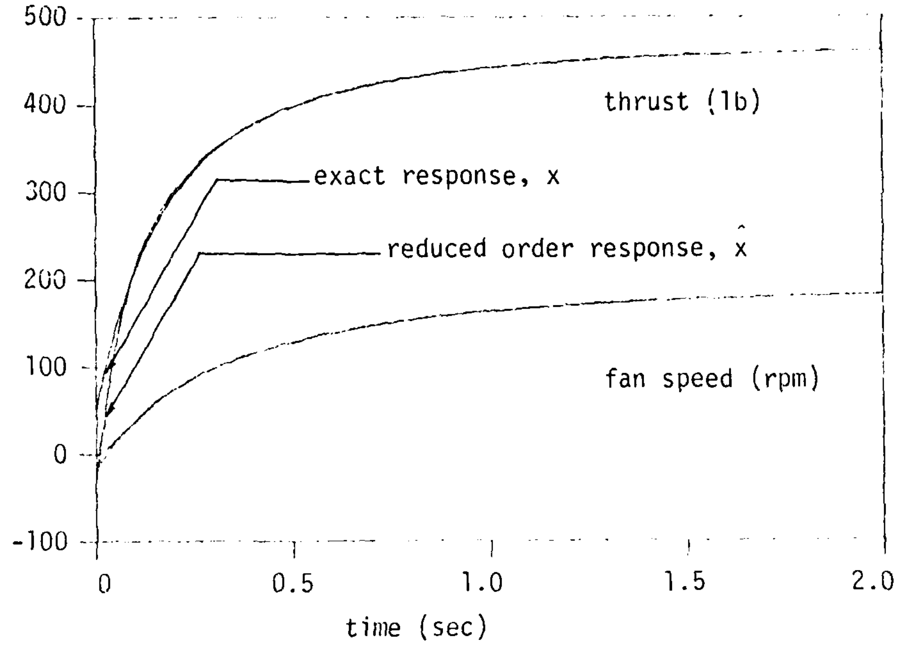

Figure 3:

Turbofan engine: Respo[illegible] of thrust and fan speed [illegible] a 500 lb/hr step change [illegible] fuel flow rate.

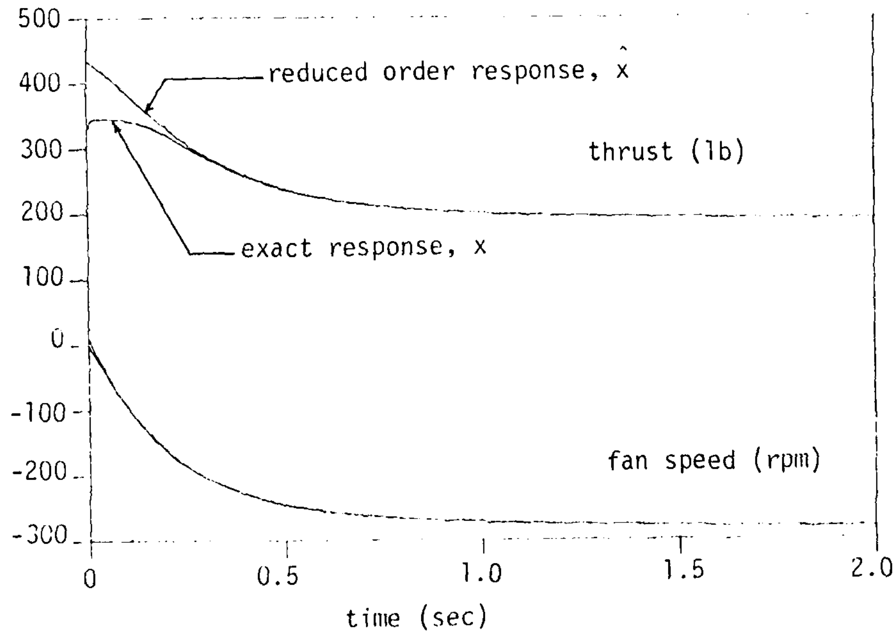

Figure 4:

Turbofan engine: Respo[illegible] of thrust and fan speed [illegible] a 10 degree step change in inlet guide vane position.

2. The time-varying problem—an exact approach.

Several earlier papers (cf. Kokotovic [16], Chow and Kokotovic [7], O'Malley and Anderson [28] and Anderson [3]) have discussed time-invariant problems, so let us now consider the time-varying system

$$(1) \qquad \dot{x} = A(t)x + B(t)u(t) \ , \qquad 0 \le t \le T \ ,$$

where A and Bu are specified.

A system such as (1) will be called two-time-scale on the interval $[0,T]$ if the spectrum $\lambda(A(t))$ of the $n \times n$ matrix A can be partitioned into two sets $S(t)$ and $F(t)$ with n_1 and $n_2 = n - n_1$ elements, respectively, such that

$$\lambda(A(t)) = S(t) \cup F(t) \ ,$$

where the eigenvalues satisfy

$$(2) \qquad \max_{s_i \in S} |s_i(t)| \equiv \tilde{s}(t) << \tilde{f}(t) \equiv \min_{f_j \in F} |f_j(t)|$$

throughout $0 \le t \le T$ with

$$(3) \qquad \mu = \max_{0 \le t \le T} (\tilde{s}(t)/\tilde{f}(t)) << 1 \ .$$

Roughly, then, μ is an upper bound for a ratio of time-varying eigenvalues. We note that if $|\mathrm{Re}\ f_j(t)|$ is large, a corresponding vector solution of the unforced system will be locally exponentially growing or decaying, while if $|\mathrm{Im}\ f_j(t)|$ is large, the corresponding solution will oscillate rapidly locally. We also note that different modelers might select different values of n_1 for the same system, and that the more difficult problems where n_1 varies across $[0,T]$ will not be considered here. Finally, the common situation where the eigenvalues of A cluster into several sets might be handled by repeated application of our technique, cf. Kokotovic *et al.* [17] and Winkelman *et al.* [40].

For general time-varying systems, it is well known that eigenvalue stability does not imply stability, cf. e.g. Coppel [8]. The result is, however, more nearly true for singularly perturbed systems. Thus, for the singularly perturbed initial value problem for

$$\dot{x} = A(t,\kappa)x + B(t,\kappa)z + C(t,\kappa) \ ,$$

$$\kappa\dot{z} = D(t,\kappa)x + E(t,\kappa)z + F(t,\kappa) \ ,$$

with smooth coefficients on $0 \leq t \leq T$, the limiting solution as κ tends to 0^+ on an interval $0 < t \leq T$ will satisfy the reduced order system

$$\dot{X} = A(t,0)X + B(t,0)Z + C(t,0) \ , \qquad X(0) = x(0) \ ,$$

$$0 = D(t,0)X + E(t,0)Z + F(t,0) \ ,$$

provided the matrix $E(t,0)$ is stable throughout $0 \leq t \leq T$. Further, an initial boundary layer (or region of nonuniform convergence) occurs in the z variable near $t = 0$ and the fast dynamics there evolve on a $\tau = t/\kappa$ time scale, cf. O'Malley [26,27]. Such theory suggests that eigenvalue stability may be appropriate for determining the behavior of two-time-scale systems. These results apply to systems where the coefficient matrices A,...,F have bounded t and κ derivatives. Related problems on the semi-infinite interval $t \geq 0$ are treated in Hoppensteadt [15], Barman [5], and Vidyasagar [36]. With less smoothness, counterexamples exist and caution must be observed, cf. Kreiss [18,19]. For these reasons, Kreiss introduced hypotheses demanding that $E(t,0)$ be "essentially diagonally dominant."

We shall not suppose that the given system (1) is two-time-scale, but rather that it can be transformed into a two-time-scale system by a time-varying transformation

$$(4) \qquad y = \mathcal{T}(t)x \ ,$$

with the system for y being "time-scale decoupled" throughout $0 \le t \le T$. Specifically, let the transformation matrix $\mathcal{T}$ have the form

$$(5) \qquad \mathcal{T}(t) = \begin{bmatrix} I_{n_1} + K(t)L(t) & K(t) \\ L(t) & I_{n_2} \end{bmatrix} = \begin{bmatrix} I_{n_1} & K(t) \\ 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ L(t) & I_{n_2} \end{bmatrix} ,$$

and let the matrices $L(t)$ and $K(t)$ be determined so that

$$(6) \qquad \dot{y} = \tilde{A}(t)y + \mathcal{T}(t)B(t)u \ ,$$

where $\tilde{A}$ has the block-diagonal form

$$\tilde{A}(t) = \begin{bmatrix} \tilde{A}_{11}(t) & 0 \\ 0 & \tilde{A}_{22}(t) \end{bmatrix}$$

with the n_1 eigenvalues of $\tilde{A}_{11}$ being small in magnitude compared to those of $\tilde{A}_{22}$ throughout $0 \le t \le T$. For $u = 0$, the slow modes for (6) would be given by $y = \begin{bmatrix} y_1 \\ 0 \end{bmatrix}$ where y_1 satisfies the lower order system $\dot{y}_1 = \tilde{A}_{11}y_1$

while the fast modes are $\begin{bmatrix} 0 \\ y_2 \end{bmatrix}$ where $\dot{y}_2 = \tilde{A}_{22}y_2$. We note that the transformation matrix T has the explicit inverse

$$(7)\qquad T^{-1}(t) = \begin{bmatrix} I_{n_1} & -K(t) \\ -L(t) & I_{n_2} + L(t)K(t) \end{bmatrix}$$

so T is always nonsingular and transformations between x and y coordinates are particularly convenient. Analogous transformations have been employed in the singular perturbations context by Wasow [38], Harris [14], and Kokotovic [16], for discrete problems by Phillips [29], and for difference equations by Matheij [21].

As a first step toward time-scale decoupling, let us set

$$(8)\qquad z = T_1(t)x$$

for the block triangular matrix

$$T_1(t) = \begin{bmatrix} I_{n_1} & 0 \\ L(t) & I_{n_2} \end{bmatrix}.$$

Clearly

$$(9)\qquad \dot{z} = \hat{A}(t)z + T_1(t)B(t)u$$

where

$$\hat{A}(t) = (T_1 A + \dot{T}_1)T_1^{-1} = (\tilde{A}_{ij})$$

$$= \begin{bmatrix} A_{11} - A_{12}L & A_{12} \\ \dot{L} + LA_{11} - A_{22}L - LA_{12}L + A_{21} & A_{22} + LA_{12} \end{bmatrix},$$

presuming the original A matrix and $\hat{A}$ are both partitioned after their first n_1 rows and columns. In order for $\hat{A}$ to be upper block-triangular, the $n_2 \times n_1$ matrix L must satisfy the matrix Riccati equation

$$\text{(10)} \qquad \dot{L} = A_{22}L - LA_{11} + LA_{12}L - A_{21}$$

throughout $0 \leq t \leq T$. Selecting $\dot{L}(e) = 0$ for a yet-unspecified endpoint $e = 0$ or T makes $\dot{T}_1(e) = 0$ and $T_1(e)$ a similarity transformation. Thus $\hat{A}(e)$ will be two-time scale provided $A(e)$ is. Let us suppose

(H1) $A(e)$ <u>has</u> n_1 <u>"slow" eigenvalues in</u> $S(e)$ <u>and</u> n_2 <u>"fast" eigenvalues in</u> $F(e)$, $e = 0$ <u>or</u> T.

This will actually determine the integers n_1 and n_2 used throughout. We'll later find that selecting $e = 0$ ($e = T$) will be natural if the fast eigenvalues of $A(t)$ are all stable (unstable) everywhere.

We now begin an extended discussion on how to compute $L(e)$. In so doing, we make improvements on previous solutions to the time-invariant A problem for which $L(t)$ is constant. If we partition the spectral decomposition of $A(e)$ as

$$A(e) = M\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}M^{-1},$$

where $\lambda(J_1) = S(e)$ and $M = (M_{ij})$, we can always reorder the entries in the state vector x so that the $n_1 \times n_1$ matrix M_{11} is nonsingular. The columns of $\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$ will span the n_1 dimensional eigenspace of $A(e)$ corresponding to the slow eigenvalues in $S(e)$ and

(11) $$L(e) = -M_{21}M_{11}^{-1}$$

will be the unique solution of the algebraic Riccati equation

(12) $$A_{22}(e)L - LA_{11}(e) + LA_{12}(e)L - A_{21}(e) = 0 ,$$

i.e. $\dot{L}(e) = 0$, achieving the time-scale decoupling

(13) $$\lambda(\hat{A}_{11}(e)) = S(e) \text{ and } \lambda(\hat{A}_{22}(e)) = F(e) .$$

Though the matrix equation (12) has many solutions, only (11) provides the desired time-scale separation (cf. Anderson [2]). We also note that (11) avoids the use of vectors in the n_2 dimensional fast eigenspace. An alternative representation

(14) $$L(e) = Q_{22}^{-1}Q_{21}$$

in terms of the left eigenspace corresponding to the n_2 fast eigenvalues of $A(e)$ would be more practical if $n_2 << n_1$. The corresponding upper triangular transformation might then be more convenient then T_1, since it would first isolate the purely slow component. Here we have partitioned $M^{-1} = (Q_{ij})$ after its first n_1 rows and columns and the invertibility of Q_{22} follows from that of M_{11}. The nontrivial result (11) follows via linear algebra, as does (14). Specifically, if $\hat{A}_{11}(e)$ has the decomposition XGX^{-1}, the algebraic Riccati equation can be rewritten as

$A_{21}(e) - A_{22}(e)L(e) = -L(e)XGX^{-1}$. For $Y = -L(e)X$, $A(e)\begin{bmatrix}X\\Y\end{bmatrix} = \begin{bmatrix}X\\Y\end{bmatrix}G$, so $\lambda(\hat{A}_{11}(e)) = S(e)$ implies that we must have $\begin{bmatrix}X\\Y\end{bmatrix} = \begin{bmatrix}M_{11}\\M_{21}\end{bmatrix}K$ for some non-singular K, i.e. $L(e) = -YX^{-1} = -M_{21}M_{11}^{-1}$. Calculating further with this $L(e)$, $\hat{A}(e) = T_1(e)A(e)T_1^{-1}(e)$ is upper block triangular. Recent work in Medanic [22] also describes the invariant manifolds of such matrix Riccati equations. Watkins [39] mentions numerical difficulties occurring when M_{11} is ill-conditioned.

Note further that any n_1 dimensional basis of the slow eigenspace could be used instead of $\begin{bmatrix}M_{11}\\M_{21}\end{bmatrix}$ in (11) to obtain $L(e)$. One possibility is to compute n_1 Schur vectors for this slow eigenspace, cf. e.g. Laub [20]. Once an approximate $L(e)$ is obtained, one may improve on its accuracy by iteration in the linear equation

(15) $$L_{i+1} = (A_{22}(e) + L_iA_{12}(e))^{-1}(L_iA_{11}(e) + A_{21}(e)) .$$

Anderson [2] shows that this iteration converges linearly with asymptotic rate $\tilde{f}^{-1}(e)\tilde{s}(e)$, so this method is particularly well-suited to systems whose time-scale separation parameter μ is very small.

The iteration scheme (15) can be obtained from the simultaneous iteration method (cf. Stewart [34] and Avramovic [4]) for calculating the dominant eigenspace corresponding to the n_2 fast eigenvalues of $A(e)$. That method generates the $n_2 \times n$ matrix V as the limit of the iteration

(16) $$V_{k+1} = V_kA(e) .$$

Splitting $V_k = [V_{k1}\ V_{k2}]$ after its first n_1 columns and setting $L_k = V_{k2}^{-1}V_{k1}$ (cf. the alternative representation (14) for $L(e)$), (16) reduces to (15). The asymptotic rate of convergence $\tilde{f}^{-1}(e)\tilde{s}(e)$ was known in this context. Indeed, the fact that (16) converges globally under very mild assumptions on V_0 implies that our iteration scheme (15) will also be robust with respect to initial iterates. Thus L_0 need not be generated through a preliminary approximate eigenanalysis of the slow eigenspace of $A(e)$; in practice a trivial L_0 achieves convergence. The reader won't be surprised to find closely related analysis in the stiff differential equations literature, cf. e.g. Alfeld and Lambert [1].

The Riccati differential equation for $L(t)$ will *have the constant* solution $L(e)$ when A is constant or when it is possible to find a constant basis $\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}$, with N_1 invertible, for the slow eigenspace of $A(t)$. *Otherwise, we need to integrate the* $n_2 \times n_1$ *dimensional* initial or terminal value problem (10), (11) for $L(t)$. We shall assume that it provides a transformed system for z which is two-time-scale. Specifically, we suppose:

(H2) the solution $L(t)$ of the matrix Riccati problem remains bounded throughout $0 \le t \le T$ and the eigenvalues of the matrix $\hat{A}_{11}(t) = A_{11} - A_{12}L$ remain small in magnitude compared to those of $\hat{A}_{22}(t) = A_{22} + LA_{12}$ throughout the interval.

If this hypothesis fails at any point, our order reduction procedure will simply not be appropriate. We note that some leeway is allowed in judging the separation of eigenvalues between $\hat{A}_{11}$ and $\hat{A}_{22}$, i.e. in deciding just how small a μ is small enough. Computational and stability aspects of the integration procedure will be illustrated below through discussion and examples.

One can proceed further and block diagonalize the upper triangular matrix $\hat{A}$ by a second nonsingular transformation

$$(17) \qquad y = T_2(t)z = T(t)x$$

for

$$T(t) = T_2(t)T_1(t)$$

and

$$T_2(t) = \begin{bmatrix} I_{n_1} & K(t) \\ 0 & I_{n_2} \end{bmatrix}$$

cf. (5). Thus (17) converts (1) into the two-time-scale system

$$(18) \qquad \dot{y} = \tilde{A}(t)y + \tilde{B}(t)u$$

cf. (6) where

$$\tilde{A}(t) = (TA + \dot{T})T^{-1} = \begin{bmatrix} \hat{A}_{11}(t) & \tilde{A}_{12}(t) \\ 0 & \hat{A}_{22}(t) \end{bmatrix}$$

with $\tilde{A}_{12} = \dot{K} - \hat{A}_{11}K + K\hat{A}_{22} + A_{12}$ and $\tilde{B} = TB$. If the $n_1 \times n_2$ matrix K satisfies the linear differential equation

$$(19) \qquad \dot{K} = \hat{A}_{11}(t)K - K\hat{A}_{22}(t) - A_{12}(t) ,$$

the matrix $\tilde{A}$ will be block diagonalized and the system for y will be time-scale decoupled, i.e. the system for the first n_1 components of y will be completely decoupled from that for its last n_2 components. Corresponding to the endpoint condition $\dot{L}(e) = 0$ for L, we now impose the condition $\dot{K}(T - e) = 0$ at the opposite endpoint because the variational equation

$$(20) \qquad \dot{\mathcal{L}} = -\hat{A}_{11}\mathcal{L} + \mathcal{L}\hat{A}_{22}$$

for L is opposite in stability to the equation (19) for K. Thus $K(T - e)$ will satisfy the Liapunov equation

$$(21) \qquad \hat{A}_{11}(T - e)K(T - e) - K(T - e)\hat{A}_{22}(T - e) - A_{12}(T - e) = 0 .$$

Its solution is unique because $\hat{A}_{11}$ and $\hat{A}_{22}$ have no common eigenvalues, cf. e.g. Bellman [6]. An explicit solution is given by $K(T - e) = -M_{12}(T - e)Q_{22}(T - e)$ where M_{12} and Q_{22} are sub-blocks of the modal matrix M for $A(T - e)$ and its inverse Q, cf. (11) and (14). It is preferable, however, to obtain $K(T - e)$ numerically by iteration in the equation

$$(22) \qquad K_{j+1}(T - e) = (\hat{A}_{11}(T - e)K_j(T - e) - A_{12}(T - e))\hat{A}_{22}^{-1}(T - e)$$

with initial iterate $K_0(T - e) = 0$. As in the iteration scheme for $L(e)$, the convergence will be rapid for $||\hat{A}_{22}^{-1}(T - e)|| \; ||\hat{A}_{11}(T - e)|| \leq \mu$

small in the spectral norm. When A is time-invariant, this provides the constant matrix K appropriate throughout the interval. More generally, however, we assume

(H3) <u>The solution</u> $K(t)$ <u>of the linear terminal or initial value problem</u> (19), (21) <u>stays bounded throughout</u> $0 \leq t \leq T$.

We note that (H3) is automatically satisfied if $K(T - e)$ and the coefficients in the differential equation (19) remain bounded.

With these three hypotheses, our time-varying LK transformation (5) has now become completely determined, and our problem (1) is reduced to solving the time-scale decoupled system

$$\dot{y}_1 = \hat{A}_{11}(t)y_1 + \tilde{B}_1(t)u \ , \tag{23}$$

$$\dot{y}_2 = \hat{A}_{22}(t)y_2 + \tilde{B}_2(t)u \ , \tag{24}$$

where $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $\tilde{B} = TB = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}$ is partitioned after its first n_1 rows. Boundary conditions for x and y are related through the nonsingular matrix T. The solutions of (23) and (24) are given by

$$y_i(t) = Y_i(t)c_i + \int^t Y_i(t)Y_i^{-1}(s)\tilde{B}_i(s)u(s)ds \ , \tag{25}$$

$i = 1$ and 2, for constant vectors c_i, where the Y_i are fundamental matrices satisfying

$$\dot{Y}_i = \hat{A}_{ii}(t)Y_i \ , \qquad Y_i(0) = I_{n_i} \ , \qquad 0 \leq t \leq T \ .$$

Though the representation (25) is useful, direct numerical integration for y_1 and y_2 is preferable to numerical implementation of (25).

Using the spectral norm, our two-time-scale assumption implies that $||\hat{A}_{11}(t)|| = \tilde{s}(t)$ while $||A_{22}^{-1}(t)|| \leq \tilde{f}^{-1}(t)$, so $||\hat{A}_{11}(t)|| = ||\hat{A}_{11}(t)|| \; ||\hat{A}_{22}^{-1}\hat{A}_{22}|| \leq \tilde{s}(t)\tilde{f}^{-1}(t)||\hat{A}_{22}(t)|| \leq \mu||\hat{A}_{22}(t)||$. Thus, $Y_2(t)$ is rapidly varying compared to $Y_1(t)$. Indeed, when $\hat{A}_{22}(t)$ has eigenvalues with large negative real parts, say of order $O(\frac{1}{\varepsilon T})$, Y_2 decays to zero exponentially fast and it becomes negligible outside an initial $O(\varepsilon T)$ boundary layer. Likewise, when the eigenvalues of $\hat{A}_{11}(t)$ are small, like $O(\kappa)$, $Y_1(t)$ is nearly constant throughout $[0,T]$ provided $T << 1/\kappa$. It is natural, then, to think of y_1 as the predominantly slow solution and of y_2 as the predominantly fast solution, realizing that the slow/fast interpretation could be corrupted by the forcing control $\tilde{B}(t)u(t)$. This slow/fast decomposition would carry back to the original system (1) as

$$x(t) = T^{-1}(t)\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} . \tag{26}$$

Altogether, then, we've transformed our original problem (1) under hypotheses (H1)-(H3) into the integration of four separate problems for L, K, y_1 and y_2, with L and K being constant for time-invariant A matrices. We'll now show how the procedure can be substantially simplified through approximations when we impose a fast mode stability assumption. Other approximations will be appropriate under different hypotheses.

3. Reduced order modeling for the initial value problem—approximate analysis.

Let's now consider the initial value problem for (1), assuming that the time scale parameter (cf. (3) and (H1)) is small, i.e.

(27) $$\mu \ll 1 ,$$

and that

(H4) the eigenvalues $f_j(t)$ of $\hat{A}_{22}(t)$ all have large negative real parts throughout $0 \leq t \leq T$.

Then

(28) $$\varepsilon \equiv -\left\{T \max_{\substack{1 \leq j \leq n_2 \\ 0 \leq t \leq T}} \left[\operatorname{Re} f_j(t)\right]\right\}^{-1} \ll 1$$

also holds.

Because (3) implies that $||\hat{A}_{11}|| \ll ||\hat{A}_{22}||$, we can expect the solution $\tilde{\mathcal{L}}$ of the linear variational equation (20) for L to be well-approximated through the nearby system $\dot{\tilde{\tilde{L}}} = -\tilde{\tilde{L}}\hat{A}_{22}$. Further, the large magnitude and stability of the eigenvalues of $\hat{A}_{22}$ suggest, via singular perturbations theory, that the initial value problem for $\tilde{\mathcal{L}}$ will have bounded solutions asymptotic to $\dot{\mathcal{L}}(t) = 0$ away from $t = 0$, while the solution of the corresponding terminal value problem will become unbounded for $t < T$. Therefore, errors made in the numerical integration of the Riccati equation for $L(t)$ should decay exponentially to zero in forward time and grow exponentially in reverse time. To keep the calculated $L(t)$ bounded, then, under hypothesis (H4), we must take $e = 0$, i.e. we define L and K through initial and terminal value problems, respectively. Indeed, the linear system for K will be well-approximated through the nearby system $\dot{\tilde{K}} = -\tilde{K}\hat{A}_{22} - A_{12}$ since $\mu << 1$, and as $\varepsilon \to 0$ (and $||\hat{A}_{22}|| \to \infty$) the limiting solution will satisfy $\dot{K} \simeq 0$ for $t < T$. Thus, the familiar quasi-steady state approximation, consistent with our terminal condition $\dot{K}(T) = 0$, holds asymptotically. For this reason, we rewrite the system (19) for K as

$$K(t) = K_1(t) + S(K(t)) \tag{29}$$

with the nonhomogeneous term

$$K_1(t) = -A_{12}(t)\hat{A}_{22}^{-1}(t)$$

and the linear operator

$$S(K) = (\hat{A}_{11}K - \dot{K})\hat{A}_{22}^{-1} .$$

We shall solve the system by successive approximations, starting with the trivial iterate $K_0(t)$ 0. Thus, we successively define the approximants

$$K_j = K_1 + \sum_{\ell=1}^{j} S^{\ell}(K_1) , \qquad j \geq 2 , \tag{30}$$

for K where $S^{\ell}(K_1) = S(S^{\ell-1}(K_1))$ for each $\ell \geq 1$ and $S^0(K_1) = K_1$. In practice, only a few iterates will be needed because $S(K)$ has a small norm due to the sizes of μ and ε, i.e. of $||\hat{A}_{11}||$ $||\hat{A}_{22}^{-1}||$ and $||\hat{A}_{22}^{-1}||$. This iteration scheme avoids the need to directly integrate the terminal value problem for K and to store its solution for later use in evaluating T and T^{-1} and for integrating the initial value problem for y_1. The successive differentiations of K_1 involved don't pose a real problem because $\dot{K}(t)$ is asymptotically negligible for $t < T$. Indeed, if we omit the derivative term in (29), our iterates (30) at $t = T$ coincide with those of (22) used to obtain $K(T)$. The resulting slow-mode or quasi-steady state approximation $K_s(t)$ to $K(t)$ will be asymptotically valid for $t < T$. The approximation $K_s(t) \simeq K(t)$ should even be fairly good near $t = T$, because we picked $\dot{K}(T) = 0$.

Returning, then, to the initial value problem (24) for y_2, with $y_2(0) = \left[L(0) \;\; I_{n_2}\right] x(0)$, the fact that $\mathrm{Re}\,\lambda(\hat{A}_{22})$ has only large stable elements suggests that y_2 should be nearly equal to its slow-mode (or quasi-steady state) approximation $\dot{y}_2 \simeq 0$, i.e.

$$(31) \qquad y_{2s}(t) \equiv -\hat{A}_{22}^{-1}(t)\tilde{B}_2(t)u(t)$$

for $t > 0$. Indeed, singular perturbations theory would show that the "composite" solution

$$(32) \qquad \tilde{y}_2(t) \equiv y_{2s}(t) + y_{2f}(t)$$

will provide a uniformly valid approximation to y_2, with the fast-varying vector y_{2f} satisfying

$$(33) \qquad \dot{y}_{2f} = \hat{A}_{22}(t)y_{2f}\,, \qquad y_{2f}(0) = y_2(0) - y_{2s}(0)$$

and decaying to zero exponentially in an $O(\varepsilon T)$ neighborhood of $t = 0$. [If a good approximation to y_2 is needed near $t = 0$, it is necessary to integrate the system for y_{2f} only over a short initial interval, but *with a relatively small mesh spacing.)* Because $\tilde{y}_2$ satisfies

$$\dot{\tilde{y}}_2 = \hat{A}_{22}\tilde{y}_2 + \tilde{B}_2 u + \dot{y}_{2s}\ , \qquad \tilde{y}_2(0) = y_2(0)$$

comparison with (24) suggests that the composite vector $\tilde{y}_2$ will be a good approximation to y_2 (and y_{2s} will be a good approximation to y_2 away from $t = 0$) provided $|\dot{y}_{2s}|$ is small on $0 \leq t \leq T$ compared, say, to the supremum of $|y_2(0)|$ and $|y_{2s}(t)|$. Thus, we'll assume

(H5) <u>the slow-mode approximation</u> y_{2s} <u>to</u> y_2 <u>is slowly-varying throughout</u> $0 \leq t \leq T$.

We recall that slowly-varying functions play an important role in asymptotic analysis (cf. Feshchenko <u>et al</u>. [13]) and note that the assumption is reasonable in the common situation that y_{2s} is itself small when $|\tilde{B}_2 u|$ is small compared to the large $||A_{22}||$. Hypothesis (H5) also reflects the fact that rapid variation of $\tilde{B}_2 u$ could cause the state y_2 to be fast for $t > 0$, even though the free response would be asymptotically negligible there. We note, in particular, that because $\hat{B}_2 = [L \quad I_{n_2}]B$, y_{2s} could become rapidly varying when our asymptotics break-down because $\dot{L}$ isn't small or the forcing Bu is rapid. The asymptotic decomposition (32) of y_2 into slow and fast parts could also be motivated by using Laplace's method (cf. Olver [25]) on the integral representation (25) for y_2.

We shall integrate the full n_1 dimensional system (23) for y_1 using the initial vector $y_1(0) = (I_{n_1} + K(0)L(0) \quad K(0))x(0)$. (If the eigenvalues of $\hat{A}_{11}$ have large real parts, we might also be able to approximate

y_1 by a quasi-steady state approximation y_{1s} within (0,T). Then, however, a change of time scale $s = \lambda t$ for an appropriate constant λ might eliminate this stiffness.) When $||\hat{A}_{11}||$ isn't large and the control term $\tilde{B}_1 u$ isn't rapidly-varying, (23) can be integrated with step-sizes much larger than would be necessary for integrating the original system.

For $t > 0$, then, the solution of our original problem will be well-approximated by the slow-mode approximation

$$\hat{x}(t) = T^{-1}(t)\begin{bmatrix} y_1(t) \\ y_{2s}(t) \end{bmatrix} . \tag{34}$$

(If desired, we'd have to correct this near $t = 0$ by taking the fast-mode correction $y_{2f}(t)$ to y_{2s} into account.) For $t > 0$, we've achieved a substantial order reduction because we need only integrate an initial value problem for $L(t)$ and another for $y_1(t)$. This is because y_{2s} is obtained explicitly from the algebraic equation (31) and K is obtained from a fast-converging iteration scheme (30), under our fast-mode stability assumption.

All these arguments can be made completely rigorous by explicitly using the small parameters μ and ε to rescale our differential equations and carrying out a careful asymptotic analysis as ε and μ simultaneously tend toward zero. For only moderately small parameters, a full integration of the linear systems for K and y_2 might be needed. The connection between our approximations and numerical methods for systems of stiff differential equations is closest to the smooth approximate particular solution technique of Dahlquist [9] and Oden [24].

To summarize, we list the somewhat oversimplified steps appropriate for obtaining a reduced order model of our fast-mode stable, two-time scale system on $t > 0$. They are:

(1) Use the eigenvalues of $A(0)$ to determine the number n_1 of slow modes.

(2) Obtain $L(0)$ by iterating in the equation
$L_{i+1} = (A_{22}(0) + L_i A_{12}(0))^{-1}(L_i A_{11}(0) + A_{21}(0))$,
$i \geq 0$, $L_0 = 0$.

(3) Integrate the initial value problem for
$\dot{L} = A_{22}L - LA_{11} + LA_{12}L - A_{21}$, making sure that the transformed system remains fast-mode stable and two-time scale with n_1 constant throughout $0 \leq t \leq T$ and that the slow-mode approximation y_{2s} remains slowly-varying there.

(4) Obtain $K(t)$ on $[0,T]$ through the iteration
$K_{j+1}(t) = (-A_{12}(t) + \hat{A}_{11}(t)K_j(t) - \dot{K}_j(t))\hat{A}_{22}^{-1}(t)$,
$j \geq 0$, $K_0(t) = 0$. (Alternatively, obtain the slow-mode approximation $K_s(t)$ by omitting the derivative term.)

(5) Integrate the initial value problem for $y_1(t)$ and obtain the reduced-order solution

$$x(t) = \begin{bmatrix} I_{n_1} \\ -L(t) \end{bmatrix} y_1(t) + \begin{bmatrix} -K(t) \\ I_{n_2} + L(t)K(t) \end{bmatrix} y_{2s}(t) \text{ for } t > 0.$$

4. Related Problems.

a. Nearly constant slow modes are found for two-time-scale systems when the eigenvalues of $\hat{A}_{22}$ are not large. Then, the small size of the eigenvalues of A_{11} suggests that $Y_1(t)$ is nearly constant on a fixed finite interval. Though the dynamics for y_2 are not simplified, we obtain order reduction in the sense that y_1 will simply track its forcing, i.e. $y_1(t) \approx y_1(0) + \int_0^t B_1(s)u(s)ds$.

b. It may sometimes be simpler to simply block-triangularize our system matrix through the matrix T_1. In the system (9) for $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, the fast modes are decoupled via $L(t)$, so a slow-mode approximation for z_2 could be used in the forcing for z_1 on $t > 0$. We block-diagonalized our system, since the linear problem for $K(t)$ seems simple after the quadratic problem for $L(t)$.

c. When the eigenvalues of $\hat{A}_{22}$ have both large positive and large negative real parts, the initial (terminal) value problem for $L(t)(K(t))$ will no longer be well-posed. Only certain two-point problems for x can be expected to have bounded solutions (cf. O'Malley [26] and O'Malley and Anderson [28]). Effective use of time-scale separation should, nonetheless, be computationally significant in obtaining approximate solutions to appropriate two-point problems.

d. The n_1 "slow" solutions of the unforced problem are spanned by the columns of the matrix $\begin{pmatrix} I_{n_1} \\ -L(t) \end{pmatrix} Y_1(t)$. If we therefore integrate the initially slow modes of our system (1) forward in time to obtain the $n \times n_1$ matrix $\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix}$, we'll necessarily have $L(t) = -X_2(t)X_1^{-1}(t)$. Thus existence of L is guaranteed as long as $X_1(t)$ remains nonsingular. For problems where L becomes unbounded on $0 < t < T$, there still remains the possibility of reinitializing our problem to keep the appropriate $n_1 \times n_1$ matrix nonsingular. This corresponds to the reorthonormalizations used by Scott and Watts [32].

5. Numerical Examples.

In practice, the need for reduced-order modelling requires us to use our schemes on problems where the time-scale separation parameter μ and the fast-mode stability parameter ε are not asymptotically small. Among many other considerations, we must then be particularly concerned with the difference between eigenvalue stability and actual stability and with the occurrence of eigenvalues with large imaginary parts that can allow slow modes (so classified by eigenvalue magnitudes) to decay faster than some fast-modes. The latter concern might be illustrated through a third order system with the slow eigenvalues $s = -1$ and the fast eigenvalues $f_{1,2} = -0.1 \pm i10$. Then, $\varepsilon = \frac{10}{T}$, so for T sufficiently large, all modes will be negligible away from $t = 0$. Otherwise, the fast modes cannot be ignored. A check on the relative stability of the slow and fast subsystems can be made through the ratio $\sigma(t) = -\max_i |Re(s_i(t))| / \max_j (Re\ f_j(t))$ and its maximum over $0 \leq t \leq T$.

If $A(t)$ has the time-varying spectral decomposition $A = MJM^{-1}$, the change of variables $w = M^{-1}x$ converts the problem (1) into $\dot{w} = (J - M^{-1}\dot{M})w + M^{-1}Bu$. Thus eigenvalue rotation, measured by the size of $M^{-1}\dot{M}$, can substantially alter the stability suggested by the eigenvalues of A and J. Rapid variation of the slow-eigenspace of A could, in particular, make L and y_{2s} rapidly-varying, and jeopardize the appropriateness of our approximations.

We shall consider two time-varying third order problems with one slow mode. Specifically, let the state matrices $A_i = MJ_iM^{-1}$ for $i = 1$ and 2 have

$$M = \begin{pmatrix} -1 & -1 & -2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 + h(t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1 + h(t))^{-1} \end{pmatrix}$$

with J_1 and J_2 being real canonical forms with spectra $\lambda(J_1) = -(1 + h'(t))\{1,10,12\}$ and $\lambda(J_2) = -(1 + h'(t))\{1,3 \pm 10i\}$. Thus for both examples, $\mu \approx 0.1$ while $(\varepsilon_1,\varepsilon_2) = (0.022,0.074)$ for $T = 4$. Therefore, the fast-mode stability and the relative stability of the fast modes is stronger for the first example. We'll also take $x(0) = B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $u(t) = \sin^2 \pi t$, and $h(t) = h'(t) = \frac{1}{8} \sin \pi t/4$.

For both examples, the appropriate initial condition for the Riccati differential equation (10) is the two vector $L(0) = \begin{pmatrix} \ell_1(0) \\ \ell_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Since the quadratic equation (12) provides a steady-state for the corresponding differential equation (10) at $t = 0$, we might attempt to find L(0) as an equilibrium solution. Figure 5 represents the $\ell_1 - \ell_2$ phase plane for example one. As shown, all points above the line $\ell_1 = 3\ell_2 - 1$ converge to L(0), but points below this line diverge to infinity. For example two, a slow oscillatory convergence is illustrated in Figure 6. Thus, this natural way to seek L(0) is only locally convergent, in contrast to the safer, globally convergent iterative method we described previously via (15).

Once L(0) is obtained, the time-varying Riccati equation (10) can be integrated from t = 0 to 4. The solution $L(t) = \begin{pmatrix} \ell_1(t) \\ \ell_2(t) \end{pmatrix}$ for example one is illustrated in Figure 7. We have also plotted the smooth solution $\tilde{L}(t)$ with $\tilde{L}(0) = L(0)$ of the algebraic Riccati equation obtained when we set the derivative term in (10) to zero. $\tilde{L}(t)$ is a good approximation to L(t). This should not be unexpected since $E(t) = L(t) - \tilde{L}(t)$ satisifes $\dot{E} = (A_{22} + \tilde{L}A_{12})E - E(A_{11} - A_{12}\tilde{L}) + EA_{12}E - \dot{\tilde{L}}$ on $0 \leq t \leq 4$ with $E(0) = 0$. Presuming $A_{22} + \tilde{L}A_{12}$ maintains large, strongly stable eigenvalues compared to $A_{11} - A_{12}\tilde{L}$ and presuming $\dot{\tilde{L}}$ isn't large, singular perturbations would suggest a small error E(t) throughout $0 \leq t \leq 4$. Thus, we could often expect to use $\tilde{L}$, the solution of an algebraic system, to approximate L(t). Figure 7 also includes trajectories $\hat{L}(t)$ for the Riccati system with perturbed initial conditions. They, too, converge to L(t) for t > 0 provided the initial perturbations aren't too large. With the weaker fast-mode stability of Example 2, we found that the initial behavior of $\hat{L}$ trajectories was oscillatory and convergence to L was delayed (cf. Figure 8).

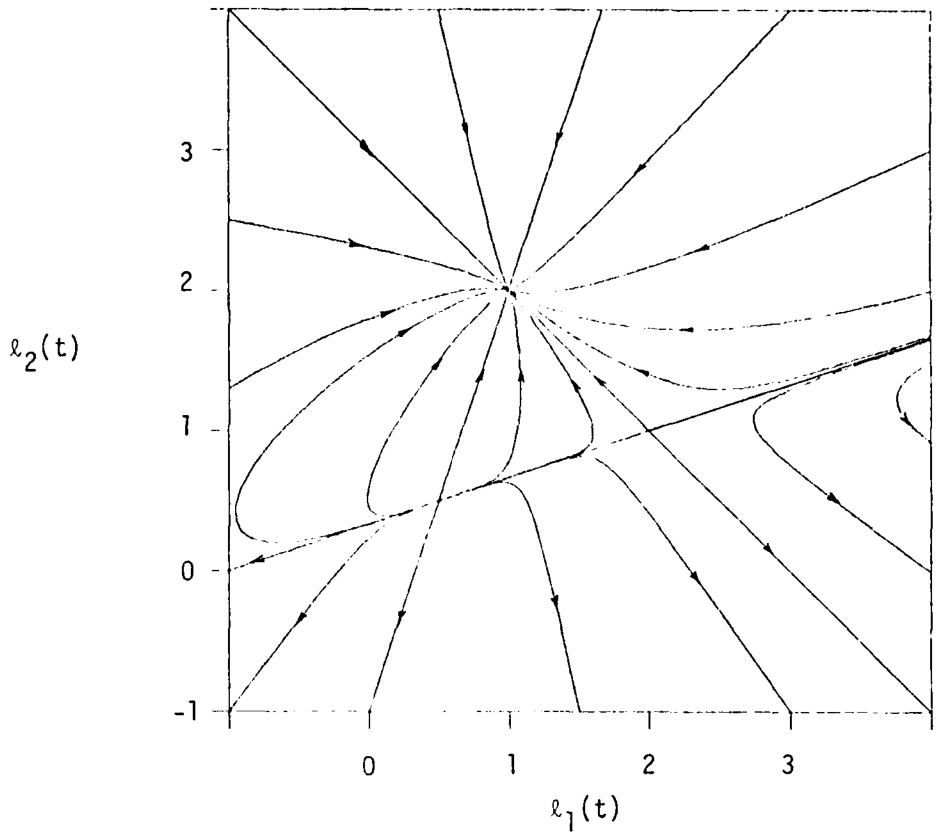

Figure 5:

The phase plane for the Riccati solution components for Example 1.

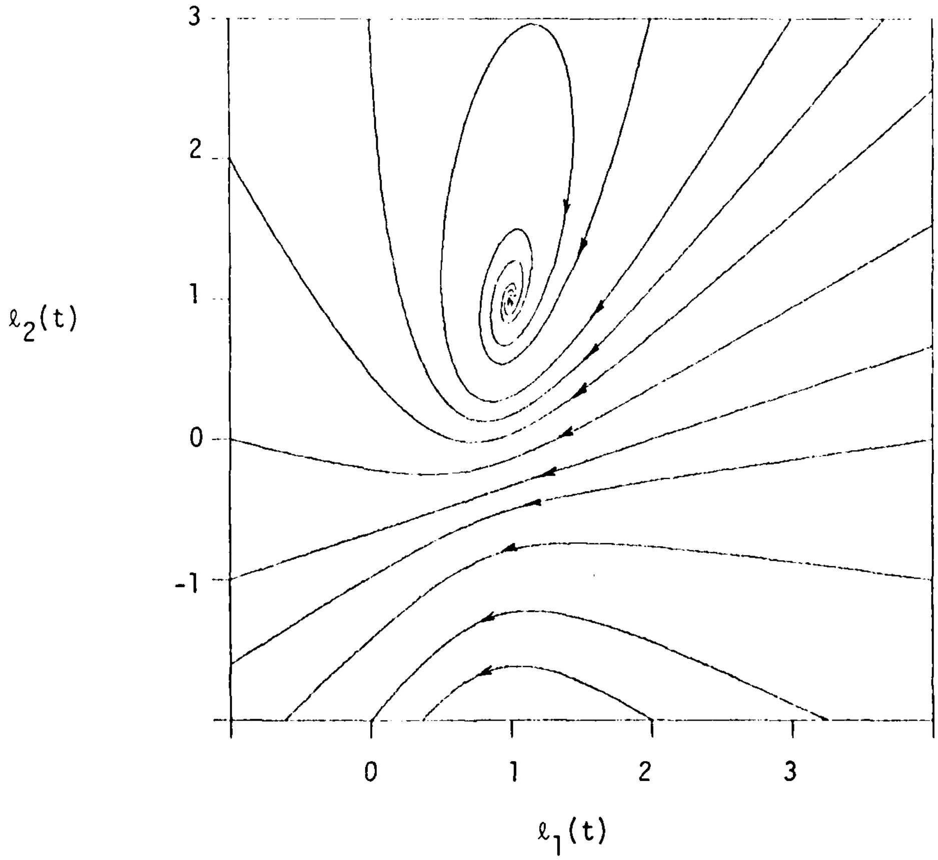

Figure 6:

The phase plane for the Riccati solution components for Example 2.

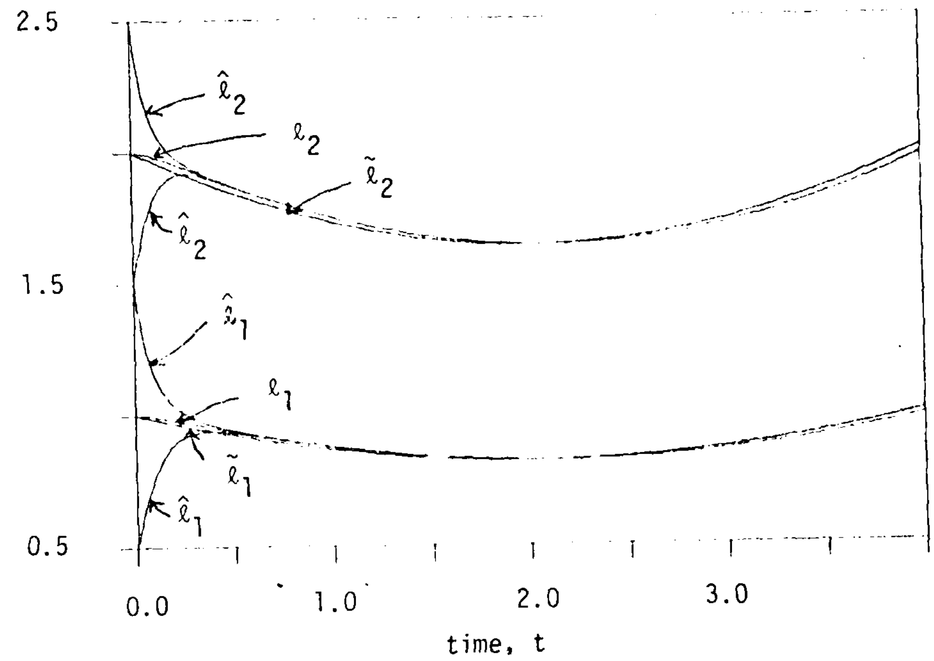

Figure 7:

Solutions L(‘
and $\tilde{L}(t)$ of
the Riccati
differential
equation and
the algebraic
Riccati
equation for
Example 1.

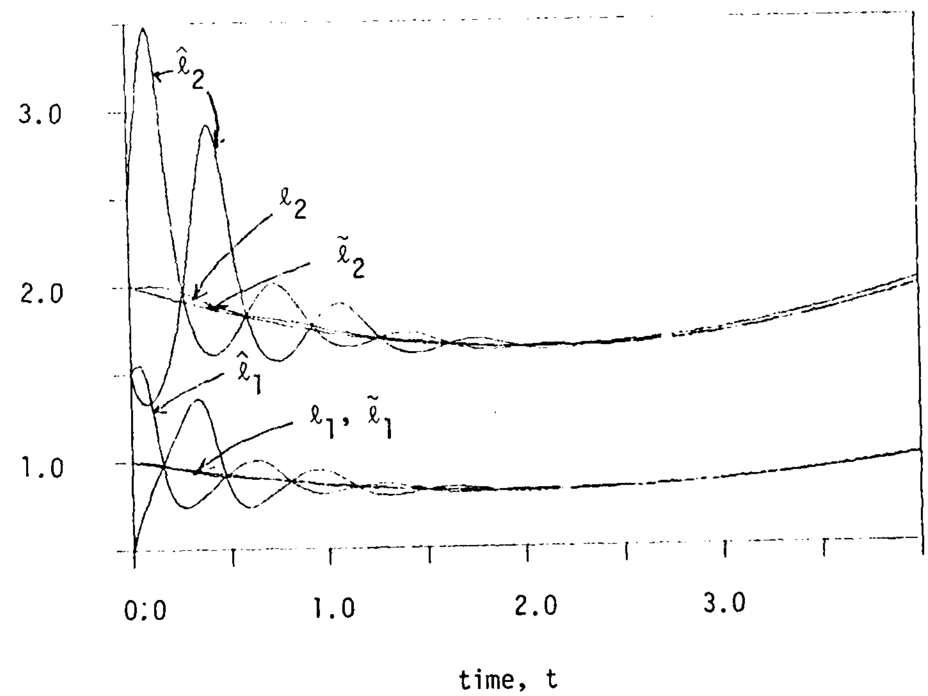

Figure 8:

Solutions L(
and $\tilde{L}(t)$ of
the Riccati
differential
equation anc
the algebrai
Riccati
equation for
Example 2.

The linear differential system (19) for $K(t) = (k_1(t) \;\; k_2(t))$ was integrated backward in time from $t = 4$ to $t = 0$. It could also be solved readily via the iteration approach. The relative behavior of $K(t)$, of the corresponding linear algebraic problem for $\ddot{K}(t)$, and of the problem for $\hat{K}(t)$ with perturbed terminal values is analogous to that for $L(t)$, $\tilde{L}(t)$, and $\hat{L}(t)$, except that the convergence of $\hat{K}$ to K also holds for large perturbations of end vectors.

By changing h' to the more oscillatory $\frac{1}{8}\sin \pi t$, the eigenvalues of A are changed, but there is little change in the solution $L(t)$ of the Riccati system. The corresponding change of $h(t)$ to $\frac{1}{8}\sin \pi t$, however, produces a more rapid oscillation in the eigenspace of $A(t)$ and there results more rapid change in the decoupling vector $L(t)$. Nonetheless, as Figures 9 and 10 illustrate, $\tilde{L}(t)$ still remains a good approximation to $L(t)$.

By superposition, the solution $x(t)$ of our forced initial value problem (1) can be considered to be the sum of the separate responses of the unforced system with initial vector $x(0)$ and to the input $u(t)$ with zero initial state. The unforced response is illustrated in Figures 11 and 12. The exact solution $x(t)$ is well approximated by the first order approximation $\hat{x}(t)$ outside an initial transient region of approximate thickness 20ε. This corresponds to 0.5 time units for Example 1 and 1.7 units for Example 2. For slowly-varying control inputs $u(t)$, the agreement between $\hat{x}(t)$ and $x(t)$ is very good for both examples. For the rapidly varying control input $u(t) = \sin^2 \pi t$, however, the resulting approximations $\hat{x}(t)$ for Example 1 are better than for Example 2, which has weaker fast-mode stability (cf. Figures 13 and 14).

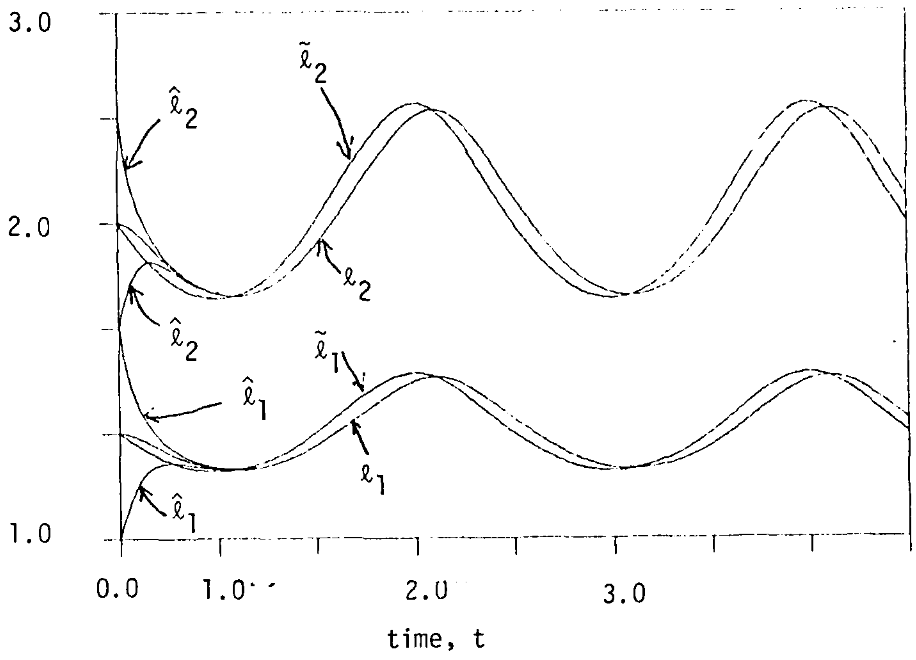

Figure 9:

Riccati solutions for Example 1 and a more oscillatory h'(t).

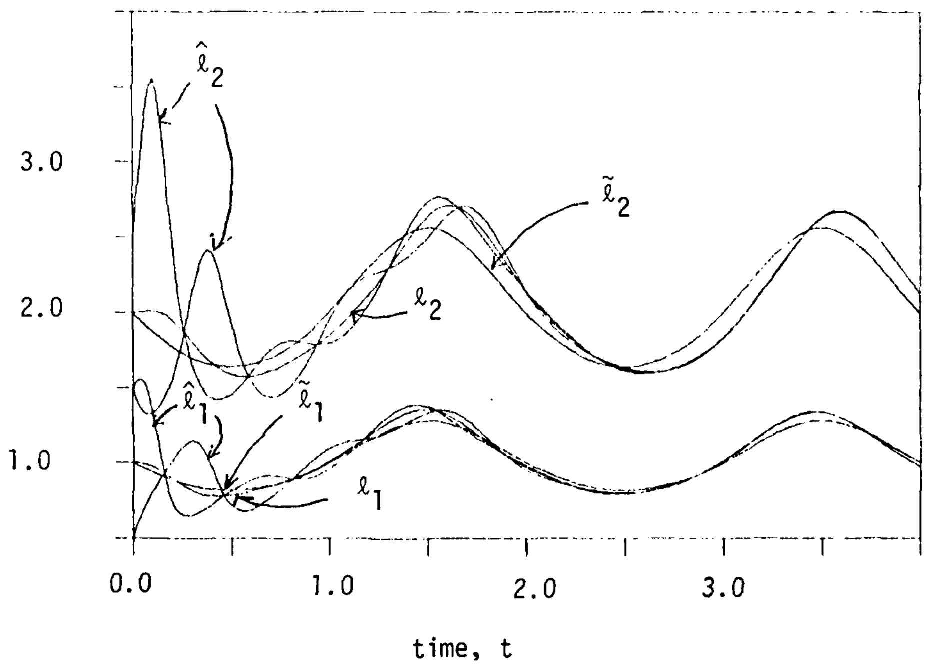

Figure 10:

Riccati solutions for Example 2 and new h'(t).

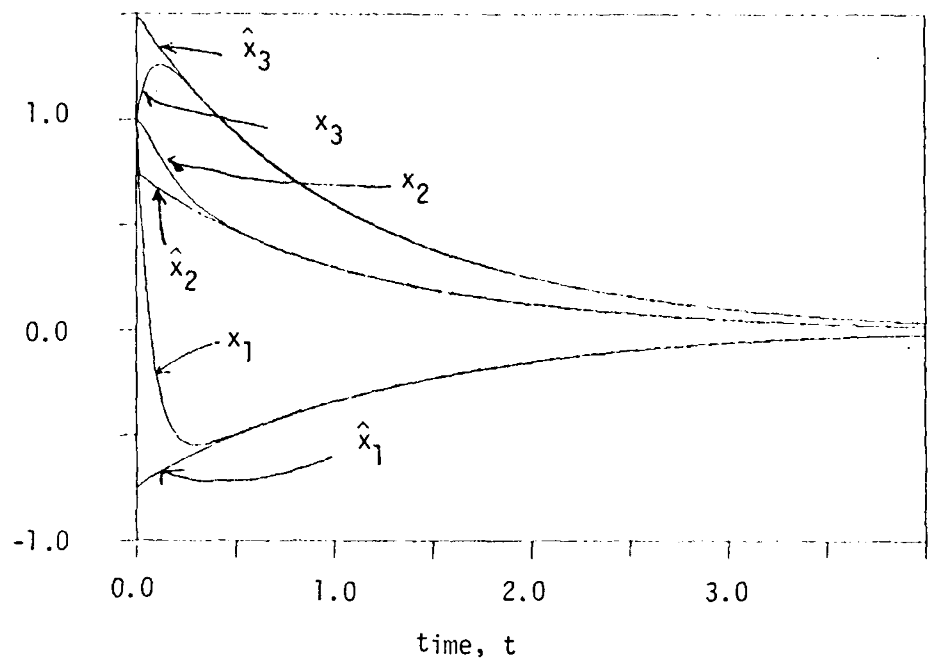

Figure 11:

Response of Example 1 with u(t) = 0

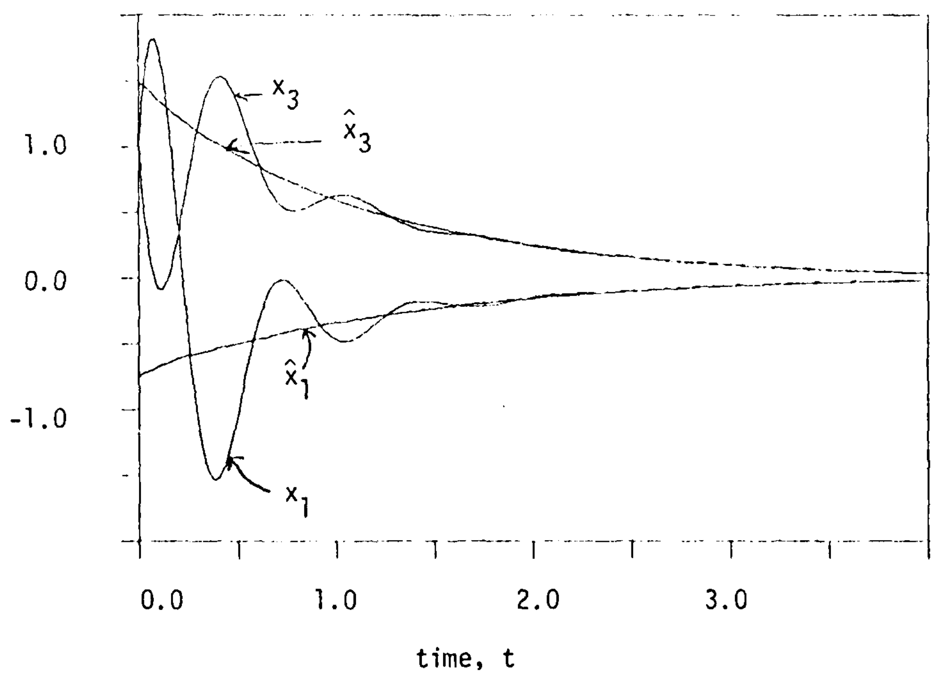

Figure 12:

Response of Example 2 wi[illegible] u(t) = 0.

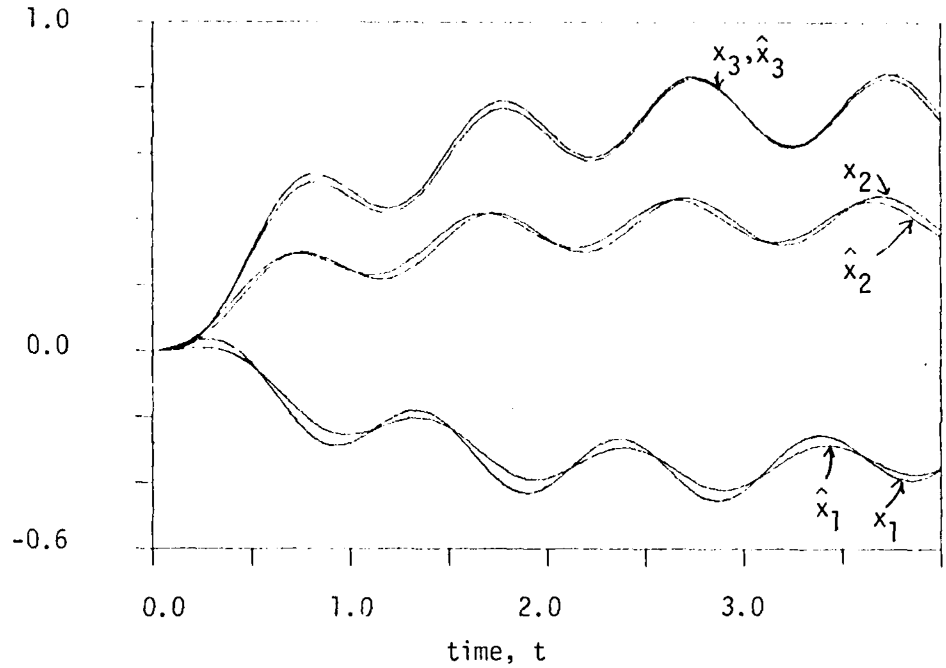

Figure 13:

Response of Example 1 to control $u(t) = \sin^2 \pi t$ and initial state $x(0) = 0$.

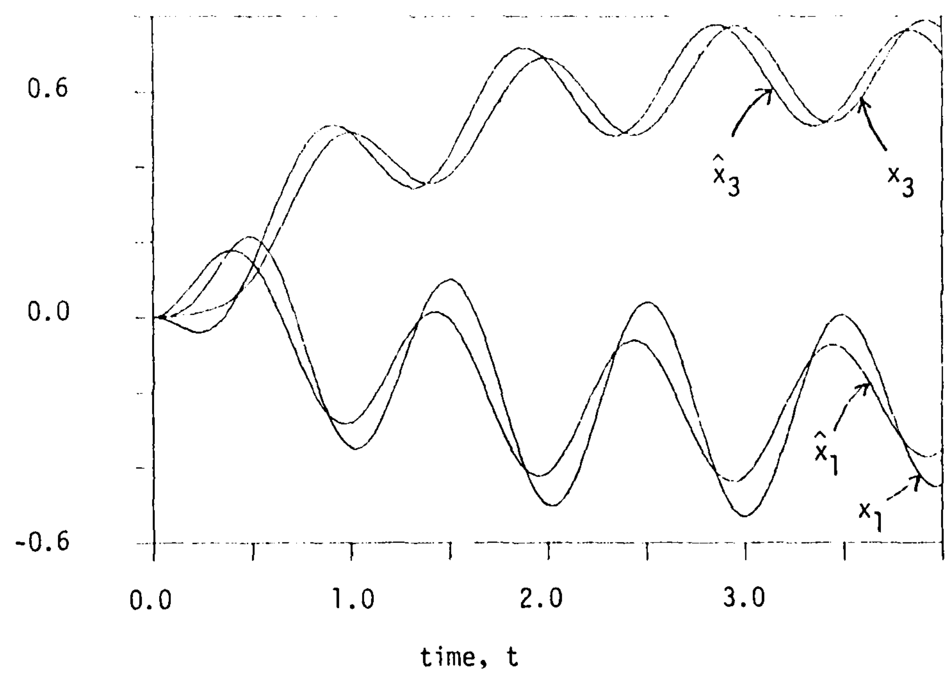

Figure 14:

Response of Example 2 to control $u(t) = \sin^2 \pi$ and initial state $x(0) =$

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The class of time-varying linear systems which are two-time-scale may be decoupled by a time-varying transformation into separate subsystems containing the slow and fast dynamic parts. The transformation is obtained by solving a nonsymmetric Riccati differential equation forward in time and a linear matrix equation backward in time. Small parameters measure the strength of the time scale separation and the fast subsystem stability. As these parameters go to zero, the order of the system is reduced and a useful approximate solution is obtained. The transformation is illustrated for examples.

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